Local and global extrema

Identify all the local maxima and minima (if they exist) for the given function, and ascertain if each local extremum qualifies as a global extremum.

$$f(x,y) = -x^3 + 2xy + y^2 + x.$$

Solution

The conditions for the first derivative are:

$$f'_x(x,y) = -3x^2 + 2y + 1 = 0$$
$$f'_y(x,y) = 2x + 2y = 0$$

Solving the second equation gives us $y = -x$, which transforms the first equation into $3x^2 + 2x - 1 = 0$, leading to two solutions $x = \frac{1}{3}, x = -1$. This results in two pairs satisfying the first derivative conditions: $\left(\frac{1}{3}, -\frac{1}{3}\right)$ and $(-1, 1)$.

The Hessian matrix of f is:

$$\begin{bmatrix} -6x & 2 \\ 2 & 2 \end{bmatrix}$$

We can verify the conditions for minima and maxima by examining whether the Hessian matrix is positive definite or negative definite. A matrix is positive definite if all of its leading principal minors are positive. Conversely, a matrix is negative definite if the signs of the leading principal minors alternate, beginning with a negative. For $x = \frac{1}{3}$, the Hessian is indefinite. and it is positive definite for $x = -1$. For $\left(\frac{1}{3}, -\frac{1}{3}\right)$ the leading principal minors are:

$$D_1 = -6 \cdot \frac{1}{3} = -2 < 0$$

And

$$D_2 = -2 \cdot 2 - 2 \cdot 2 = -8 < 0$$

For $(-1, 1)$

$$D_1 = -6 \cdot -1 = 6 > 0$$

And

$$D_2 = 6 \cdot 2 - 2 \cdot 2 = 4 > 0$$

Hence:

- $\left(\frac{1}{3}, -\frac{1}{3}\right)$ is a saddle point
- $(-1, 1)$ is a local minimum.

The point $(-1, 1)$ does not represent a global minimum; for instance, the value of the function at $f(2, 0)$ which is -6 is less than $f(-1, 1)$ which is -1.